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Paper No. : Electromagnetic Theory

Module : Conservation Laws - II

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Description of Module	
Subject Name	Physics
Paper Name	Electromagnetic Theory
Module Name/Title	Conservation Laws - II
Module Id	M6

Contents

Learning Objectives:

From this module students may get to know about the following:

1. Proper understanding of the law of conservation of momentum and angular momentum.
2. Why these laws of conservation require that the electromagnetic field itself must be assigned momentum and angular momentum.

6. Conservation laws – II

This module is a continuation of the study of conservation laws for a system involving electromagnetic fields. Whereas in the last module we looked at the law of conservation of energy, in this module we study the laws of conservation of momentum and angular momentum.

6.1 Electromagnetic force and conservation of momentum

6.1.1 Electromagnetic force

We now study the conservation of linear momentum. First look at the expression for the total electromagnetic force on a charged particle:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \frac{d\vec{P}}{dt} \tag{35}$$

For a system of charged particles, if we denote the total mechanical momentum of the particles by $\vec{P}_{mech}$, then

$$\vec{F} = \frac{d\vec{P}_{mech}}{dt} = \sum_i q_i(\vec{E}_i + \vec{v}_i \times \vec{B}_i) \tag{36}$$

For a continuous charge and current distribution, we replace q_i by charge density $\rho(\vec{x})$, $q_i\vec{v}_i$ by the current density $\vec{J}(\vec{x})$ and the summation by an integration over continuous distributions; and this leads to

$$\vec{F} = \frac{d\vec{P}_{mech}}{dt} = \int_V (\rho\vec{E} + \vec{J} \times \vec{B}) d^3x \tag{37}$$

To eliminate direct reference to the sources we use Maxwell's equations to eliminate ρ and $\vec{J}$ in favour of the fields. From the equation (here we are working with the microscopic fields)

$$\vec{\nabla} \times \vec{B} = \mu_0(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}), \tag{38}$$

we have

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (39)$$

Using this and the other inhomogeneous Maxwell equation

$$\rho = \varepsilon_0 \vec{\nabla}.E$$

we simplify the integrand in (9):

$$\begin{aligned}\rho\vec{E} + \vec{J}\times\vec{B} &= \varepsilon_0 \vec{E}\,\vec{\nabla}.\vec{E} + \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\times\vec{B} \\ &= \varepsilon_0 \vec{E}\,\vec{\nabla}.\vec{E} + \varepsilon_0 \vec{B}\times\frac{\partial \vec{E}}{\partial t} - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B})\end{aligned} \qquad (40)$$

Now

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \vec{E}\times\frac{\partial \vec{B}}{\partial t} + \left(\frac{\partial \vec{E}}{\partial t}\right)\times\vec{B} = \vec{E}\times\frac{\partial \vec{B}}{\partial t} - \vec{B}\times\left(\frac{\partial \vec{E}}{\partial t}\right)$$

Or

$$\vec{B}\times\left(\frac{\partial \vec{E}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times\frac{\partial \vec{B}}{\partial t} \qquad (41)$$

On substituting this relation into (10) above, we have

$$\rho\vec{E} + \vec{J}\times\vec{B} = \varepsilon_0(\vec{E}\,\vec{\nabla}.\vec{E} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}) - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$
$$= \frac{1}{4\pi}[(\vec{E}\,\vec{\nabla}.\vec{E} - \vec{B}\times\vec{\nabla}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t} - \frac{\partial}{\partial t}(\vec{E}\times\vec{B})] \qquad (42)$$

Further, on using the Faraday's law

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

and adding the null term $\vec{B}(\vec{\nabla}.\vec{B})$, we have

$$\rho\vec{E} + \vec{J}\times\vec{B} = \varepsilon_0[\vec{E}\,\vec{\nabla}.\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})]$$
$$+\frac{1}{\mu_0}[\vec{B}(\vec{\nabla}.\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})] - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \qquad (43)$$

Putting this expression for the integrand in (9), we get

$$\vec{F} = \int_V(\rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B})d^3x$$
$$= \int_V\{\varepsilon_0[\vec{E}\,\vec{\nabla}.\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})] + \frac{1}{\mu_0}[\vec{B}(\vec{\nabla}.\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})] - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\}d^3x \qquad (44)$$

If $\vec{f}$ is the force density, i.e., the force per unit volume, then

$$\vec{f} = \varepsilon_0[\vec{E}\,\vec{\nabla}.\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})] + \frac{1}{\mu_0}[\vec{B}(\vec{\nabla}.\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})] - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \qquad (45)$$

Let us work on the first two terms, $\vec{E}(\vec{\nabla}.\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})$ and write its ith component (here we are making full use of the tensor notation along with summation convention which often proves useful in deriving vector relations):

$$[\vec{E}\,\vec{\nabla}.\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})]_i = E_i\frac{\partial E_j}{\partial x_j} - \varepsilon_{ijk}E_j(\vec{\nabla}\times\vec{E})_k = E_i\frac{\partial E_j}{\partial x_j} - \varepsilon_{ijk}E_j\varepsilon_{klm}\frac{\partial E_m}{\partial x_l}$$
$$= E_i\frac{\partial E_j}{\partial x_j} - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})E_j\frac{\partial E_m}{\partial x_l} = E_i\frac{\partial E_j}{\partial x_j} - E_j\frac{\partial E_j}{\partial x_i} + E_j\frac{\partial E_i}{\partial x_j} \qquad (46)$$
$$= \frac{\partial}{\partial x_j}(E_iE_j - \frac{1}{2}(\vec{E}.\vec{E})\delta_{ij})$$

The other two terms are obtained by simply replacing E by B, so that

$$\varepsilon_0[\vec{E}\,\vec{\nabla}.\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})]_i + \frac{1}{\mu_0}[\vec{B}(\vec{\nabla}.\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})]_i$$
$$= \frac{\partial}{\partial x_j}[\varepsilon_0(E_iE_j - \frac{1}{2}(\vec{E}.\vec{E})\delta_{ij}) + \frac{1}{\mu_0}(B_iB_j - \frac{1}{2}(\vec{B}.\vec{B})\delta_{ij})] \qquad (47)$$

This leads to the following expression for the components of the force density

$$f_i == \frac{\partial}{\partial x_j}[\varepsilon_0(E_iE_j - \frac{1}{2}(\vec{E}.\vec{E})\delta_{ij}) + \frac{1}{\mu_0}(B_iB_j - \frac{1}{2}(\vec{B}.\vec{B})\delta_{ij})] - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \qquad (48)$$

We define the symmetric Maxwell stress tensor T_{ij}:

$$T_{ij} = [\varepsilon_0 (E_i E_j - \frac{1}{2}(\vec{E}.\vec{E})\delta_{ij}) + \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}(\vec{B}.\vec{B})\delta_{ij})] \quad (49)$$

In terms of the stress tensor the expression for the force density takes a simple form

$$f_i = \frac{\partial}{\partial x_j} T_{ij} - \varepsilon_0 \frac{\partial}{\partial t}(\vec{E} \times \vec{B})_i \quad (50)$$

Further, remember that $(\vec{E} \times \vec{H}) = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$ is the pointing vector $\vec{S}$ and $\varepsilon_0 \mu_0 = \frac{1}{c^2}$ so that finally

$$f_i = (\frac{\partial}{\partial x_j} T_{ij} - \varepsilon_0 \mu_0 \frac{\partial}{\partial t} S_i) = (\frac{\partial}{\partial x_j} T_{ij} - \frac{1}{c^2} \frac{\partial}{\partial t} S_i) \quad (51)$$

The total force on the charges inside volume V is therefore

$$F_i = \int_V \frac{\partial}{\partial x_j} T_{ij} d^3x - \frac{1}{c^2} \int_V \frac{\partial}{\partial t} S_i d^3x \quad (52)$$

The first integral can be converted into a surface integral by use of Gauss theorem as applied to tensors $(\int_V \frac{\partial}{\partial x_j} T_{ij} d^3x = \oint_S T_{ij} n_j da)$. This provides an alternative expression for the force:

$$F_i = \oint_S T_{ij} n_j da - \frac{1}{c^2} \frac{d}{dt} \int_V S_i d^3x \quad (53)$$

Often the dyadic notation $\vec{\vec{T}}$ is used to denote a tensor of rank two which is described by two indices. In that case $\vec{\vec{T}}.\hat{n}$ represents a vector, the scalar product of tensor with a vector. In this notation we have

$$\vec{F} = \oint_S \vec{\vec{T}}.\hat{n}da - \frac{1}{c^2}\frac{d}{dt}\int_V \vec{S}d^3x \qquad (54)$$

Whenever $\vec{S}$ is independent of time, the second term drops out and the electromagnetic force can be expressed entirely in terms of the stress tensor at the boundary. Physically $\vec{\vec{T}}$ is the force per unit area i.e., stress acting on the surface. Or, in terms of components, T_{ij} is the force per unit area in the ith direction on a surface oriented in the jth direction. The diagonal elements, T_{xx}, T_{yy}, T_{zz}, represent pressures and the non-diagonal elements represent the shear stresses.

6.1.2 Conservation of momentum

We now come to the conservation of momentum of the system. Remember that the force represents the rate of change of mechanical momentum of the charges. Bringing the second term in equation (53) or (54) to the left hand side, we have

$$\frac{d}{dt}P_{mech,i} + \frac{1}{c^2}\frac{d}{dt}\int_V S_i d^3x = \int_V \frac{\partial}{\partial x_j}T_{ij}d^3x = \oint_S T_{ij}n_j da \qquad (55)$$

The quantity

$$\vec{p}_{field} = \frac{1}{c^2}\vec{S} = \varepsilon_0 \vec{E}\times\vec{B}$$

can be identified with the density of momentum of the electromagnetic field. Then $\frac{1}{c^2}\int_V S_i d^3x = \varepsilon_0\int_V (\vec{E}\times\vec{B})d^3x$ is the total momentum of the electromagnetic field, $\vec{P}_{field}$, within the volume V:

$$\vec{P}_{field} = \int_V \varepsilon_0(\vec{E}\times\vec{B})d^3x$$

so that

$$\frac{d}{dt}(\vec{P}_{mech} + \vec{P}_{field}) = \oint_V \vec{\nabla}.\vec{T}d^3x = \oint_S \vec{T}.\hat{n}da \qquad (56)$$

In terms of components

$$\frac{d}{dt}(P_{mech} + P_{field})_i = \int_V \frac{\partial}{\partial x_j}T_{ij}d^3x = \oint_S T_{ij}n_j da \qquad (57)$$

This equation is the statement of conservation of angular momentum. The term $T_{ij}n_j$ is the ith component of the flow per unit area of momentum across the surface S into the volume V.

If $\vec{p}_{mech}$ is the density of mechanical momentum and $\vec{p}_{field}$ that of electromagnetic field momentum, then the differential form of the law of conservation of momentum is

$$\frac{\partial}{\partial t}(\vec{p}_{mech} + \vec{p}_{field})_i = \frac{\partial}{\partial x_j}T_{ij} \qquad (58)$$

Evidently, $-T_{ij}$ is the momentum flux density, playing the role of current density $\vec{J}$ in the continuity equation or energy flux density in the Poynting theorem. Specifically, $-T_{ij}$ is the momentum in the ith- direction crossing the surface in the jth direction per unit area per unit time.

Notice that the Poynting vector $\vec{S}$ plays two different roles. Whereas $\vec{S}$ is the energy flow per unit area per unit time, $\frac{1}{c^2}\vec{S}$ is the momentum per unit volume stored in those fields. Similarly T_{ij} plays a double role, whereas T_{ij} is the electromagnetic stress acting on a surface, $-T_{ij}$ is the flow of momentum.

6.2 Conservation of Angular Momentum

It should not come as a surprise now that like energy and momentum, the electromagnetic field is also endowed with angular momentum. A proper understanding of the law of conservation of angular momentum demands that the electromagnetic field possess angular momentum. The derivation of the law of conservation of angular momentum proceeds on the same lines as energy and momentum.

The starting point is the same expression for the Lorentz force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \frac{d\vec{P}}{dt} \qquad (\backslash 59)$$

Or, for a continuous charge and current distribution

$$\vec{F} = \frac{d\vec{P}_{mech}}{dt} = \int_V (\rho\vec{E} + \vec{J} \times \vec{B}) d^3x \qquad (60)$$

Introducing the density of mechanical momentum, $\vec{p}_{mech}$, we have

$$\frac{\partial \vec{p}_{mech}}{\partial t} = \rho\vec{E} + \vec{J} \times \vec{B} \qquad (61)$$

The right hand side is written in term of the fields only [see equation (43)]:

$$\rho\vec{E}+\vec{J}\times\vec{B}=\varepsilon_0[\vec{E}\vec{\nabla}.\vec{E}-\vec{E}\times(\vec{\nabla}\times\vec{E})]+\frac{1}{\mu_0}[\vec{B}(\vec{\nabla}.\vec{B})-\vec{B}\times(\vec{\nabla}\times\vec{B})]-\varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

The mechanical angular momentum for a particle is defined as $\vec{r}\times\vec{p}$. For a continuous distribution just as momentum is defined as the volume integral of momentum density, $\vec{p}_{mech}$, angular momentum is defined as the volume integral of angular momentum density $\vec{l}_{mech}$ by

$$\vec{L}_{mech}=\int_V \vec{l}_{mech}d^3x\,; \qquad \vec{l}_{mech}=\vec{x}\times\vec{p}_{mech} \tag{62}$$

It follows from equation (36) that

$$\frac{\partial\vec{l}_{mech}}{\partial t}=\frac{\partial}{\partial t}(\vec{x}\times\vec{p}_{mech})=\vec{x}\times\frac{\partial}{\partial t}(\vec{p}_{mech})=\vec{x}\times(\rho\vec{E}+\vec{J}\times\vec{B})$$

$$=\vec{x}\times\{\varepsilon_0[\vec{E}\vec{\nabla}.\vec{E}-\vec{E}\times(\vec{\nabla}\times\vec{E})]+\frac{1}{\mu_0}[\vec{B}(\vec{\nabla}.\vec{B})-\vec{B}\times(\vec{\nabla}\times\vec{B})]-\varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\} \tag{63}$$

Bringing the last term to the left hand side and writing the ith component of the vector equation, we have

$$[\frac{\partial\vec{l}_{mech}}{\partial t}+\varepsilon_0\frac{\partial}{\partial t}\{\vec{x}\times(\vec{E}\times\vec{B})\}]_i$$

$$=\{\varepsilon_0\vec{x}\times[(\vec{E}\vec{\nabla}.\vec{E}-\vec{E}\times(\vec{\nabla}\times\vec{E})]+\frac{1}{\mu_0}\vec{x}\times[\vec{B}(\vec{\nabla}.\vec{B})-\vec{B}\times(\vec{\nabla}\times\vec{B})]\}_i$$

$$=\varepsilon_{ijk}x_j\{\varepsilon_0[\vec{E}\vec{\nabla}.\vec{E}-\vec{E}\times(\vec{\nabla}\times\vec{E})]+\frac{1}{\mu_0}[\vec{B}(\vec{\nabla}.\vec{B})-\vec{B}\times(\vec{\nabla}\times\vec{B})]\}_k$$

(64)

But

$$\varepsilon_0[\vec{E}\,\vec{\nabla}.\vec{E}-\vec{E}\times(\vec{\nabla}\times\vec{E})]_k+\frac{1}{\mu_0}[\vec{B}(\vec{\nabla}.\vec{B})-\vec{B}\times(\vec{\nabla}\times\vec{B})]_k$$
$$=\frac{\partial}{\partial x_l}[\varepsilon_0(E_kE_l-\frac{1}{2}(\vec{E}.\vec{E})\delta_{kl})+\frac{1}{\mu_0}(B_kB_l-\frac{1}{2}(\vec{B}.\vec{B})\delta_{kl})]=\frac{\partial T_{kl}}{\partial x_l} \quad (65)$$

Substituting this expression, the above equation becomes

$$[\frac{\partial l_{mech}}{\partial t}+\varepsilon_0\frac{\partial}{\partial t}\{\vec{x}\times(\vec{E}\times\vec{B})\}]_i=\varepsilon_{ijk}x_j\frac{\partial T_{kl}}{\partial x_l}$$
$$=\varepsilon_{ijk}(\frac{\partial x_jT_{kl}}{\partial x_l}-\frac{\partial x_j}{\partial x_l}T_{kl})=\varepsilon_{ijk}(\frac{\partial x_jT_{kl}}{\partial x_l}-T_{kj})=\varepsilon_{ijk}\frac{\partial x_jT_{kl}}{\partial x_l} \quad (66)$$

The second term, $\varepsilon_{ijk}T_{kj}$ is zero since ε_{ijk} is anti-symmetric in all its indices while T_{kj} is symmetric. The right hand side can be put in a more symmetric form. Since j and k are both dummy indices they can be freely interchanged. Further ε_{ijk} is anti-symmetric in j and k. Making use of these properties, we have

$$\varepsilon_{ijk}\frac{\partial x_jT_{kl}}{\partial x_l}=\frac{1}{2}\frac{\partial}{\partial x_l}(\varepsilon_{ijk}x_jT_{kl}+\varepsilon_{ikj}x_kT_{jl})=\frac{1}{2}\frac{\partial}{\partial x_l}(\varepsilon_{ijk}x_jT_{kl}-\varepsilon_{ijk}x_kT_{jl})$$
$$=\frac{1}{2}\varepsilon_{ijk}\frac{\partial}{\partial x_l}(x_jT_{kl}-x_kT_{jl}) \quad (67)$$

Replacing the right hand side of equation (66) by the above expression

$$[\frac{\partial l_{mech}}{\partial t}+\varepsilon_0\frac{\partial}{\partial t}\{\vec{x}\times(\vec{E}\times\vec{B})\}]_i=\frac{1}{2}\varepsilon_{ijk}\frac{\partial}{\partial x_l}(x_jT_{kl}-x_kT_{jl}) \quad (68)$$

The interpretation of this equation is now clear. $\varepsilon_0(\vec{E}\times\vec{B})$ is the momentum density of the field. Therefore $\vec{l}_{field}=\varepsilon_0\{\vec{x}\times(\vec{E}\times\vec{B})\}$ is the angular momentum density in the electromagnetic field. The term on the right hand side represents the rate of flow of angular momentum into the volume V. The integral form of the law of conservation of angular momentum is obtained by taking the integral of the above expression over the volume and takes the form

$$\frac{d}{dt}(\vec{l}_{mech} + \vec{l}_{field}) = \int_V \frac{1}{2}\varepsilon_{ijk} \frac{\partial}{\partial x_l}(x_j T_{kl} - x_k T_{jl}) d^3x = \oint_S \frac{1}{2}\varepsilon_{ijk}(x_j T_{kl} - x_k T_{jl}) n_l da \qquad (70)$$

We define a third rank tensor $M_{jkl} = x_j T_{kl} - x_k T_{jl}$, which has nine independent components because of anti-symmetry of the expression under $j \leftrightarrow k$ which is made explicit by the presence of ε_{ijk}. By making use of the dyadic notation the flux of angular momentum can be described by the tensor

$$\overleftrightarrow{M} = \overleftrightarrow{T} \times \vec{x}. \qquad (71)$$

Summary

1. In this module we have tried to provide to the student a proper understanding of the law of conservation of momentum and angular momentum in the presence of electromagnetic field.
2. We explain why for the laws of conservation of momentum and angular momentum to be valid locally, the electromagnetic field itself must be assigned momentum and angular momentum.
3. We learn that the momentum density of the electromagnetic field is described in terms of a second rank tensor, whereas the angular momentum density of the field is described in terms of a third rank tensor.